Chapter 11
Continuous/Discrete FT

Chapter 12
Sampling and Aliasing in Image Reconstruction

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- **Previous sessions:**
  - MRI signal (Chap. 3)
  - Fourier transform(Chaps.9 & 10)
- **Today's content**
  - Continuous FT, properties and phase imaging
  - FT pairs
  - Discrete FT (DFT) and properties
  - Infinite/finite sampling, aliasing and Nyquist Criterion
  - Image reconstruction and DFT

Continuous FT Properties

FT is the bridge between two 'conjugate' domains

- Linearity
- Duality
- Spatial Scaling
- Space Shifting (shift theorem)
- Even/odd symmetry
- Convolution
- Derivative
- Parseval (energy)

Continuous FT Properties

- Linearity:

$$\mathcal{F}[af(x) \pm bg(x)] = aF(k) \pm bG(k)$$

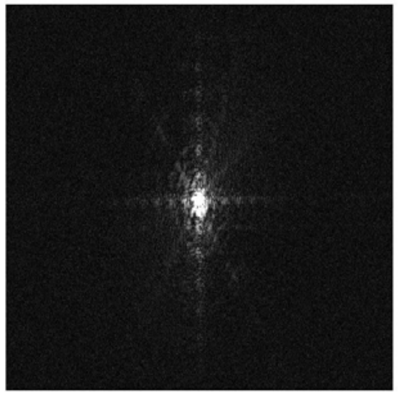
- Implications for MRI:
  - Linear calculation can be done in either k-space or image domain
  - Each point in k-space contains contributions from all excited spins

Continuous FT Properties

- Spatial Scaling:

$$\mathcal{F}[h(ax)] = \mathrm{H}(k/\mathrm{a})\ /|\mathrm{a}|$$

- Implications for MRI:
  - High spatial frequency components (smaller a) are located at high k, and vice versa
  - High spatial frequency components (smaller a) have lower energy in k-space, and vice versa. DC component, i.e. the k-space center, should have the highest energy



Continuous FT Properties

- **Shift Theorem:**

  Shift in one domain <=> Additional Linear phase in another

  e.g. $s_m(k) \rightarrow s_m(k - k_0) \Rightarrow \hat{\rho}(x) \rightarrow \hat{\rho}(x)e^{i2\pi k_0 x}$

  $s_m(k) \rightarrow s_m(k)\, e^{-i2\pi k x_0} \Leftarrow \hat{\rho}(x) \rightarrow \hat{\rho}(x - x_0)$

- **Implications for MRI:**
  - Over coverage in the k-space center is necessary to ensure a high signal baseline in the image
  - Motion/displacement of the object will introduce phase error between k-space lines, introducing artifacts (Chap.23)
  - A rigid body motion induced phase error can be recovered by adjusting the phase of corresponding k-space lines

Continuous FT Properties

- Convolution theorem

$$\mathcal{F}(g(x)\cdot h(x)) = G(k) * H(k) \square \int dk' G(k') H(k-k')$$

- Implications for MRI:
  - Important for understanding finite sampling effects in DFT
  - Filtering effects can be achieved either by multiplying the filter in image domain or convoluting with the FT form of the filter in k-space

Continuous FT Properties

- Conjugate symmetry (partial FT)

$$\text{Real}[\mathcal{F}(f(x))] = \text{Real}[\mathcal{F}(-f(x))]$$
$$\text{Imag}[\mathcal{F}(f(x))] = -\text{Imag}[\mathcal{F}(-f(x))]$$

- Implications for MRI:
  - Partially collected k-space may still contains all the information of the object (w/ conditions)
  - Uncollected k-space can be zero padded, or estimated based on the collected portion
  - Example: Partial Fourier Imaging

More Continuous FT Properties

- Duality

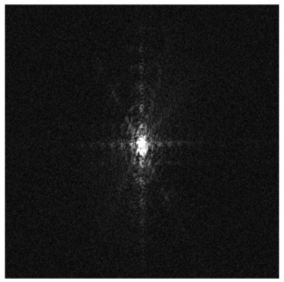
$$h(x) \overset{\mathcal{F}}{\Leftrightarrow} H(k) \qquad => \qquad \mathrm{H}(-x) \overset{\mathcal{F}}{\Leftrightarrow} h(k)$$

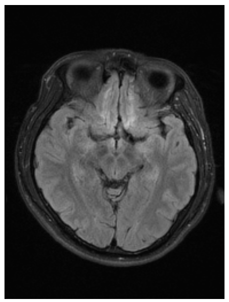
- Derivative Theorem (e.g. edge finding)

$$\mathcal{F}(f'(x)) = \mathrm{i}2\pi k \mathcal{F}(f(x))$$

- Parseval (constant energy)

$$\int_{-\infty}^{\infty} dx |h(x)|^2 = \int_{-\infty}^{\infty} dk |H(k)|^2$$





Important FT pairs

- Dirac delta function (impulse function)

$$\delta(k-k_0)=\int_{-\infty}^{\infty}dx\ e^{-i2\pi(k-k_0)x}=\mathcal{F}[e^{i2\pi k_0 x}]$$

$$\delta(x-x_0)=\int_{-\infty}^{\infty}dk\ e^{-i2\pi k(x-x_0)}=\mathcal{F}^{-1}[e^{-i2\pi k x_0}]$$

- Rectangular Function

$$\mathcal{F}\left[rect\left(\frac{x}{W}\right)\right]=W sinc(\pi W k)$$

- Gaussian

$$\mathcal{F}\left[e^{-\pi x^2}\right]=e^{-\pi k^2}$$

Signal sampling function

$$u(k) = \Delta k \sum_{p=-\infty}^{\infty} \delta(k - p\Delta k) = \Delta k \sum_{p=-\infty}^{\infty} \mathcal{F}[e^{i2\pi p\Delta kx}]$$

$$\Downarrow$$

$$U(x) = \mathcal{F}^{-1}\,[u(k)] = \Delta k \sum_{p=-\infty}^{\infty} e^{i2\pi p\Delta kx}$$

$$\Downarrow$$

$$\textstyle\sum_{n=-\infty}^{\infty} e^{i2\pi na} = \sum_{m=-\infty}^{\infty} \delta(a-m)$$

$$\delta(ax) = \delta(x)/|a|$$

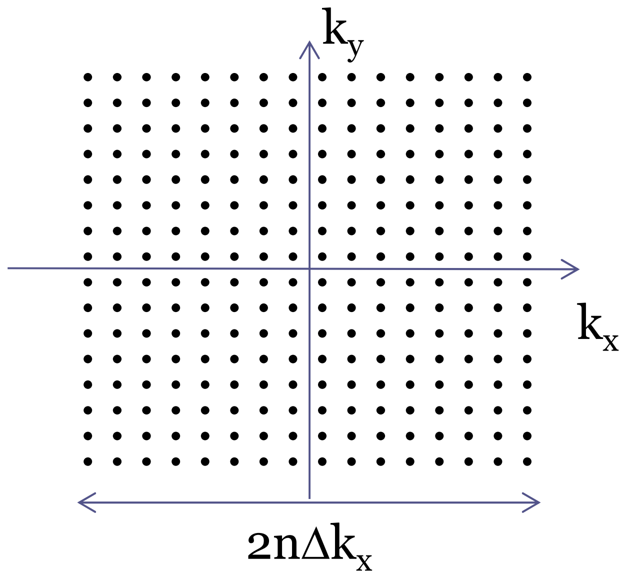
$$U(x) = \sum_{q=-\infty}^{\infty} \delta(x - \frac{q}{\Delta k})$$

Discrete FT (DFT)

- Approximation of Continuous FT
- More practical for MRI

$$G(p\Delta k) = \sum_{q=-n}^{n-1} g(q\Delta x) e^{-i2\pi q\Delta x p\Delta k}$$

where $\Delta k = 1/L$ and $L = 2n\Delta x$



DFT properties

- Linearity
- Spatial Scaling
- Space Shifting (shift theorem)
- Even/odd symmetry
- Convolution
- Derivative
- Parseval

Sampling, Aliasing and Image Reconstruction

- MRI signal is the finite, truncated, digitized version of the continuous signal picked up by the receive coil
- Discrete sampling is done using ADC (analog-digital converter)
- The sampling must meet certain criteria in order to faithfully represent the original continuous signal
- Failure in meeting the criteria will lead to artifacts
- Physical spin density $\rho(x)$ of the object is real, while reconstructed $\hat{\rho}(x)$ is practically complex

Infinite sampling

- Sampling function:

$$u(k) = \Delta k \sum_{p=-\infty}^{\infty} \delta(k - p\Delta k)$$

$$U(x) = \sum_{q=-\infty}^{\infty} \delta(x - q/\Delta k)$$

- Sampled signal:

$$s_\infty(k) \equiv s(k)u(k)$$

$$= \Delta k \sum_{p=-\infty}^{\infty} s(p\Delta k)\delta(k - p\Delta k)$$

- Spin density after DFT

$$\hat{\rho}_\infty(x) = \rho(x) * U(x)$$

$$= \sum_{q=-\infty}^{\infty} \rho(x - q/\Delta k)$$

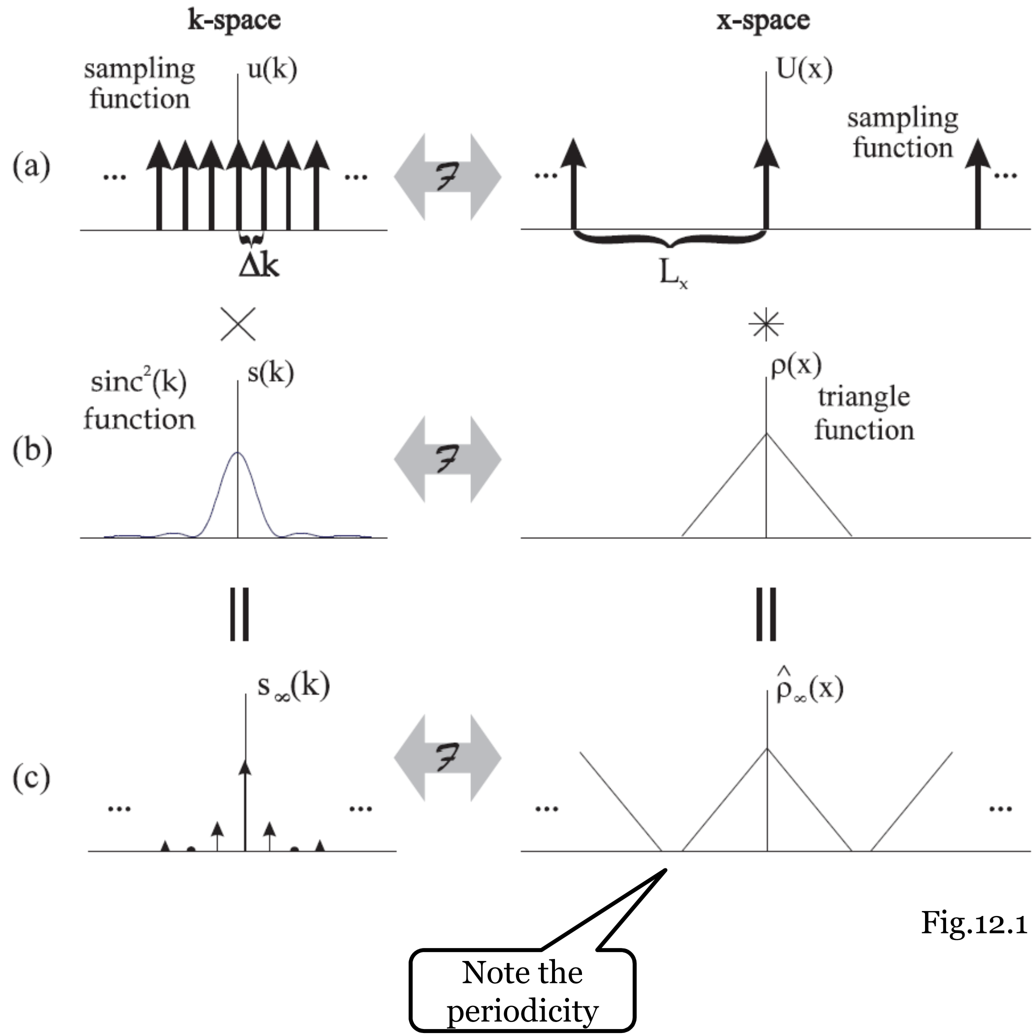


Fig.12.1

Aliasing and Nyquist Sampling Criterion

- $1/\Delta k$ L
- If object length (A) >L, then overlaps in $\hat{\rho}_\infty(x)$ will take place, introducing aliasing artifacts
- To avoid aliasing:
  A<L or $\Delta k<1/A$

Nyquist Sampling Criterion for MRI

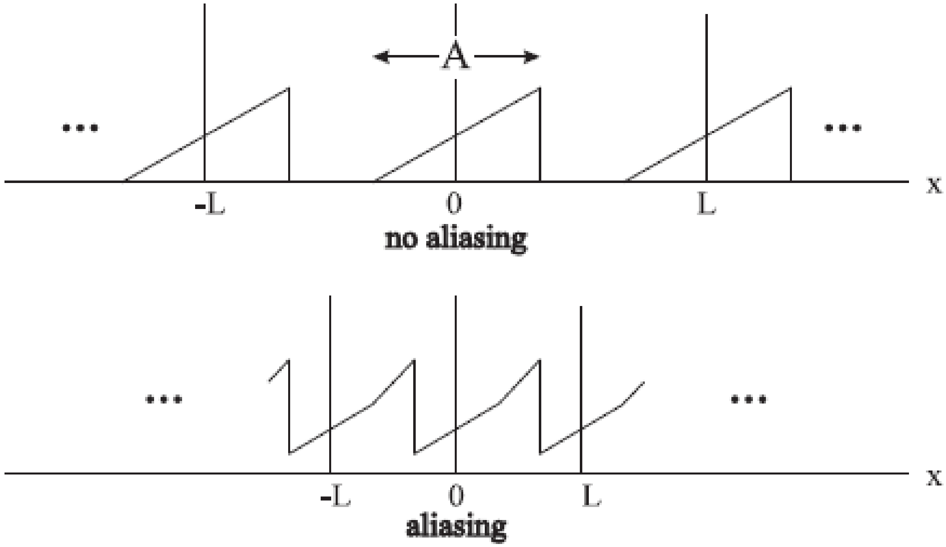
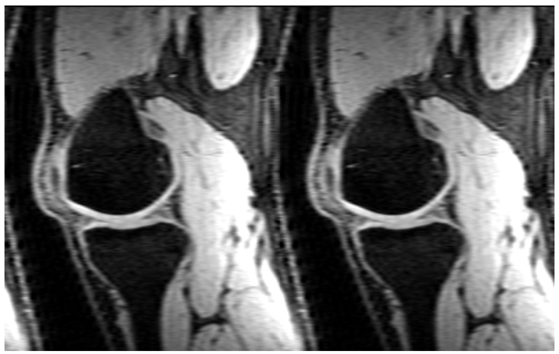


Fig.12.3



Nyquist in Read direction

$$\left.\begin{array}{c}\Delta k_R = \gamma G_R \Delta t \\ \Delta k_R = 1/L_R \\ \Delta t = T_s/N = 1/BW_R\end{array}\right\} BW_R = \gamma G_R L_R > \gamma G_R A_R$$

What we usually do is use fixed T_s and GR
but 2x oversampling, i.e. $\Delta t/2$,along Read direction

$$BW_R = \gamma G_R L_R = \frac{N}{T_s}$$

Nyquist in Phase Encoding direction

$$\left.\begin{aligned}\Delta k_{PE} &= \gamma \Delta G_{PE}\tau_{PE}\\ \Delta k_{PE} &= 1/L_{PE}\\ \Delta t = TR &= 1/BW_{PE}\end{aligned}\right\} \quad \gamma \Delta G_{PE}\tau_{PE} = \frac{1}{L_{PE}} < \frac{1}{A_{PE}}$$

What we usually do in MRI is use smaller $\Delta G_{PE}\tau_{PE}$

Counterpart in Read direction is smaller $G_R \Delta t_s$

Simply collecting more points along PE direction using fixed $\Delta G_{PE}\tau_{PE}$:

1) Will not change total bandwidth or L_{PE}

$$BW_{PE} = \frac{N}{T_{sPE}} = \frac{N}{NTR} = \frac{1}{TR}$$

2) But increase the PE resolution

$$\Delta L_{PE} = \frac{L_{PE}}{\mathrm{N}} = \frac{1}{\mathrm{N}\Delta k_{PE}}$$

Finite, discrete sampling

- Data truncation

$$s_{m(k)} = s(k) \cdot u(k) \cdot rect(\frac{k+\Delta k/2}{W})$$

$$\hat{\rho}(x) = \rho(x) * U(x) * Wsinc(\pi W x)e^{-i\pi x\Delta k}$$

Sinc shaped low-pass filter

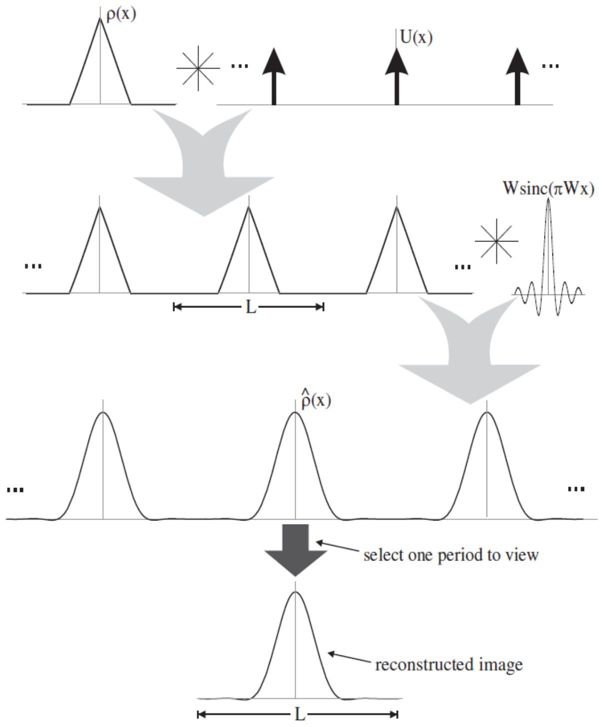


Fig. 12.6

Finite sampling and k-space symmetry

- The question: why $rect(\frac{k+\Delta k/2}{W})$ instead of $rect(\frac{k}{W})$?
- Symmetric (a) vs. MRI standard (b)
  - K-space center (i.e. k=0) and edge not well defined vs. well defined
  - For large 2n, image magnitude will be almost the same
  - A phase shift difference in image phase

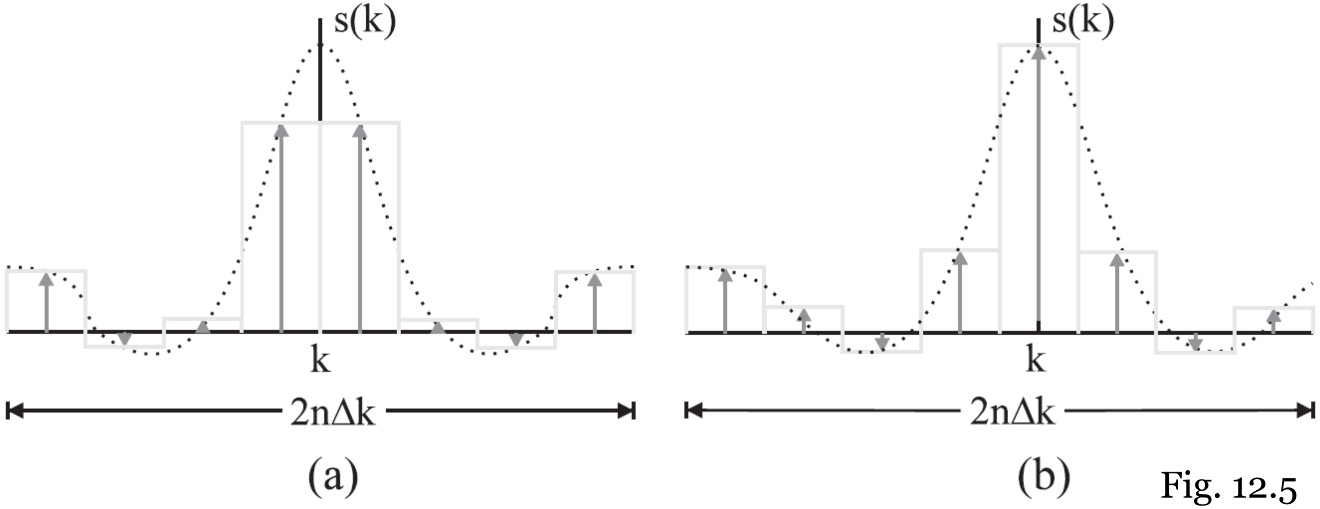


Fig. 12.5

Discrete Fourier Transform(DFT)

$$s(p\Delta k) = \Delta x \sum_{q=-n}^{n-1} \hat{p}(q\Delta x) e^{-i\pi pq/n}$$

$$\hat{p}(q\Delta x) = \Delta k \sum_{q=-n}^{n-1} s(p\Delta k) e^{i\pi pq/n}$$

- Resolution (in voxel size)

$$\Delta L_{RO} = \frac{BW_R}{N_R \gamma G_R}$$

$$\Delta L_{PE} = \frac{1}{N_{PE} \gamma \Delta G_{PE} \tau_{PE}}$$

Homework

- Prob *11.1 - 11.3, 11.8, 12.2, 12.7, 12.8*

Next Session

Chapter 13.1-13.3